

DETERMINANTAL REPRESENTATIONS OF HYPERBOLIC CURVES VIA POLYNOMIAL HOMOTOPY CONTINUATION

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ABSTRACT. A smooth curve of degree d in the real projective plane is hyperbolic if its ovals are maximally nested, i.e. its real points contain $\lfloor \frac{d}{2} \rfloor$ nested ovals. By the Helton-Vinnikov Theorem, any such curve admits a definite symmetric determinantal representation. We use polynomial homotopy continuation to compute such representations numerically. Our method works by lifting paths from the space of hyperbolic polynomials to a branched cover in the space of pairs of symmetric matrices.

INTRODUCTION

Let $p \in \mathbb{C}[t, x, y]$ be a homogeneous polynomial of degree $d \geq 1$. A *(linear symmetric) determinantal representation* of p is an expression

$$p = \det(tM_1 + xM_2 + yM_3)$$

where M_1, M_2, M_3 are complex symmetric matrices of size $d \times d$. Determinantal representations of plane curves are a classical topic of algebraic geometry. Existence in the general case, at least for smooth curves, was first proved by Dixon in 1902 [2]. For an exposition in modern language, see Beauville [1].

Real determinantal representations of real curves have only been studied systematically much later in the work of Dubrovin [3] and Vinnikov [12]. Of particular interest here are the *definite representations*, where some linear combination of the matrices M_1, M_2, M_3 is positive definite. By a celebrated result due to Helton and Vinnikov [6], these correspond exactly to the *hyperbolic curves*, whose real points consist of maximally nested ovals in the real projective plane.

The Helton-Vinnikov theorem (previously known as the Lax Conjecture) has attracted attention in connection with semidefinite programming, since it characterizes the convex subsets of the real plane that can be described by linear matrix inequalities. See Vinnikov [14] for an excellent survey.

While the Helton-Vinnikov theorem ensures the existence of a definite determinantal representation for any hyperbolic curve, finding such a representation for a given polynomial p remains a difficult computational problem. With a suitable choice of coordinates, we can restrict to representations of the form

$$p = \det(tI_d + xD + yR)$$

where I_d is the identity matrix, D is a real diagonal and R a real symmetric matrix. The hyperbolicity of p is reflected in the fact that for any point $(u, v) \in \mathbb{R}^2$, all

roots of the univariate polynomial $p(t, u, v) \in \mathbb{R}[t]$ are real. Given such p , the computational task of finding the unknown entries of D and R leads, in general, to a zero-dimensional system of polynomial equations. However, as d grows, this direct approach quickly becomes infeasible in practice. This, as well as symbolic methods and an alternative approach via theta functions based on the proof of the Helton-Vinnikov theorem, have been investigated in [10]. As far as actual computations are concerned, $d = 6$ was the largest degree for which computations terminated in reasonable time.

In this note, we present a more sophisticated numerical approach, implemented using the *NumericalAlgebraicGeometry* package [7] in MACAULAY2 [4] (NAG4M2). We consider the branched cover of the space of homogeneous polynomials by pairs of matrices (D, R) (with D diagonal and R symmetric) via the determinantal map $(D, R) \mapsto \det(tI_d + xD + yR)$. We use known results on the number of equivalence classes of complex determinantal representations to show that the determinantal map is unramified over the set of smooth hyperbolic polynomials (Thm. 1.3). We then use the fact that this set is path-connected. In fact, an explicit path connecting any hyperbolic polynomial to a certain fixed polynomial was constructed by Nuij in [9], which we refer to as the N-path. Our algorithm works by constructing a lifting of the N-path to the covering space. The advantage over just solving the zero-dimensional system of equations directly via homotopy continuation is that we need only track a single path, instead of one path for each solution. We also recover the Helton-Vinnikov theorem from these topological considerations and the count of equivalence classes of complex representations.

At this point, we do not know whether the N-path indeed avoids the singular locus. We prove this for the case $d = 2$ by considering the corresponding discriminant (Prop. 2.2). At any rate, since the singular locus has codimension at least 2 inside the set of strictly hyperbolic polynomials, the N-path can be expected to avoid singularities for almost all of the starting hyperbolic polynomials.

To account for the unlikely scenario that the (real) N-path goes through the singular locus, we provide an algorithm that produces a complex determinantal representation via the complexification of the N-path. The fact that this representation is real with a non-zero probability gives a straightforward probabilistic algorithm for finding a real determinantal representation.

Our current proof-of-concept implementation uses standard double floating point precision and evaluation of polynomials in the top-level interpreted language of MACAULAY2. Even with these limitations we can compute some examples for $d = 7$ within one day. With arbitrary precision arithmetic and speeding up the numerical evaluation procedure, we see no obstacles to computing robustly for d in double digits using the present-day hardware.

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1. DETERMINANTAL MAPPING

For $d \geq 1$, consider the affine space of polynomials

$$\begin{aligned}\mathcal{F} &= \{p \in \mathbb{C}[t, x, y] \mid p \text{ is homogeneous of total degree } d \text{ and } p(1, 0, 0) = 1\} \\ \mathcal{F}_{\mathbb{R}} &= \mathcal{F} \cap \mathbb{R}[t, x, y]\end{aligned}$$

and the vector space of pairs of complex resp. real symmetric matrices

$$\begin{aligned}\mathcal{M} &= \{(D, R) \in (\text{Sym}_d(\mathbb{C}))^2 \mid D \text{ is diagonal}\} \\ \mathcal{M}_{\mathbb{R}} &= \mathcal{M} \cap (\text{Sym}(\mathbb{R}))^2.\end{aligned}$$

The dimensions of these four spaces (over \mathbb{C} resp. \mathbb{R}) are all the same, namely $\frac{d(d+3)}{2}$. We study the map

$$\Phi: \begin{cases} \mathcal{M} & \rightarrow \mathcal{F} \\ (D, R) & \mapsto \det(tI_d + xD + yR). \end{cases}$$

and its restriction to $\mathcal{M}_{\mathbb{R}}$. A polynomial $p \in \mathcal{F}_{\mathbb{R}}$ is called *hyperbolic* (with respect to $(1, 0, 0)$) if all roots of the univariate polynomial $p(t, u, v) \in \mathbb{R}[t]$ are real, for all $(u, v) \in \mathbb{R}^2$. It is called *strictly hyperbolic* if the roots of $p(t, u, v)$ are real and distinct, for all $(u, v) \in \mathbb{R}^2$. Write

$$\mathcal{H} = \{p \in \mathcal{F}_{\mathbb{R}} \mid p \text{ is hyperbolic}\}.$$

The image of $\mathcal{M}_{\mathbb{R}}$ under Φ is contained in \mathcal{H} . It is also not hard to show that it is closed (see [11, Lemma 3.4]). Our first goal is to find a connected open subset U of \mathcal{H} such that the restriction of Φ to $\Phi^{-1}(U)$ is smooth.

Proposition 1.1.

- (1) *The set \mathcal{H} is closed in $\mathcal{F}_{\mathbb{R}}$, contractible and path-connected.*
- (2) *The interior $\text{int}(\mathcal{H})$ of \mathcal{H} is the set of strictly hyperbolic polynomials and \mathcal{H} is the closure of $\text{int}(\mathcal{H})$.*
- (3) *A polynomial $f \in \mathcal{H}$ is strictly hyperbolic if and only if the projective variety $\mathcal{V}_{\mathbb{C}}(f)$ defined by f has no real singular points.*
- (4) *Let \mathcal{H}° be the set of hyperbolic polynomials $f \in \mathcal{H}$ for which $\mathcal{V}_{\mathbb{C}}(f)$ is smooth. Then $\text{int}(\mathcal{H}) \setminus \mathcal{H}^{\circ}$ has codimension at least 2 in $\mathcal{F}_{\mathbb{R}}$.*

Proof. (1) and (2) are proved by Nuij [9] (see also below). (3) is proved in [11, Lemma 2.4]. (4) follows from the fact that the elements of $\text{int}(\mathcal{H})$ have no real singularities, while complex singularities must come in conjugate pairs. \square

For fixed $p \in \mathcal{H}$, the group $\text{SL}_d(\mathbb{C})$ acts on the determinantal representations $p = \det(tM_1 + xM_2 + yM_3)$ via symmetric equivalence. In other words, any $A \in \text{SL}_d(\mathbb{C})$ gives a new representation $p = \det(tAM_1A^T + xAM_2A^T + yAM_3A^T)$. When we restrict to the normalized representations we are considering, we have an action on pairs $(D, R) \in \Phi^{-1}(p)$ by those elements $A \in \text{SL}_d(\mathbb{C})$ for which $AA^T = I_d$ (i.e. $A \in \text{SO}_d(\mathbb{C})$) and ADA^T is diagonal.

Theorem 1.2. *Any $p \in \mathcal{F}$ has only finitely many complex representations $p = \det(tI_d + yD + zR)$ up to symmetric equivalence. If the curve $\mathcal{V}_{\mathbb{C}}(p)$ is smooth, the number of equivalence classes is precisely $2^{g-1} \cdot (2^g + 1)$, where $g = \binom{d-1}{2}$ is the genus of $\mathcal{V}_{\mathbb{C}}(p)$.*

Proof. For smooth curves, the equivalence classes of symmetric determinantal representations are in canonical bijection with ineffective even theta characteristics; see [10, Thm. 2.1] and references given there. \square

Theorem 1.3. *The set \mathcal{H}° of smooth hyperbolic polynomials is an open, dense, path-connected subset of \mathcal{H} , and each fibre of Φ over a point of \mathcal{H}° consists of exactly $2^{g-1} \cdot (2^g + 1) \cdot 2^{d-1} \cdot d!$ distinct points.*

Proof. Let $p \in \text{int}(\mathcal{H})$ and let $(D, R) \in \Phi^{-1}(p)$, which means $p = \det(tI_d + xD + yR)$. The diagonal entries of D are the zeros of $p(t, -1, 0)$. Since p is strictly hyperbolic, these zeros are real and distinct. So D is a real diagonal matrix with distinct entries. It follows then that the centralizer of D in $\text{SO}_d(\mathbb{C})$ consists precisely of the 2^d diagonal matrices with entries ± 1 . Let S be such a matrix with $S \neq \pm I$. We want to identify the set of symmetric matrices R that commute with S . Up to permutation, we may assume that the first k diagonal entries of S are equal to -1 and the remaining $d-k$ are equal to 1 . It follows then that any R with $SR = RS$ must have $r_{ij} = r_{ji} = 0$ if $i > k \geq j$, so that R is block-diagonal. For such R to show up in a pair $(D, R) \in \Phi^{-1}(p)$, the polynomial p must be reducible. In particular, if $p \in \mathcal{H}^\circ$, there is no such S commuting with R . It follows then that $\{SRS \mid S \text{ diagonal with } S^2 = I\}$ has 2^{d-1} distinct elements. Permuting the distinct diagonal entries of D gives $d!$ possible choices of D . This, combined with the count of equivalence classes in the preceding theorem, completes the proof. \square

Corollary 1.4. *The restriction of Φ to $\Phi^{-1}(\mathcal{H}^\circ)$ is smooth.*

Proof. The restriction of Φ to $\Phi^{-1}(\mathcal{H}^\circ)$ is a polynomial map with finite fibres that is unramified over \mathcal{H}° , since the cardinality of the fibre does not change. Hence it is smooth (see for example Hartshorne [5, III.10]). \square

Corollary 1.5 (Helton-Vinnikov Theorem). *Every hyperbolic curve possesses a definite real determinantal representation.*

Proof. Since all fibres of Φ over \mathcal{H}° have the same cardinality and \mathcal{H}° is path-connected, the number of real points in each fibre must also be constant over \mathcal{H}° . That number cannot be zero, since there clearly exist fibres with real points. It follows that \mathcal{H}° is contained in $\Phi(\mathcal{M}_{\mathbb{R}})$. On the other hand, $\Phi(\mathcal{M}_{\mathbb{R}})$ is closed in $\mathcal{F}_{\mathbb{R}}$ by [11, Lemma 3.4] and contained in \mathcal{H} , hence $\Phi(\mathcal{M}_{\mathbb{R}}) = \mathcal{H}$. \square

Remark 1.6. The number of equivalence classes of real definite representations of a hyperbolic curve is in fact also known, namely it is 2^g . See [10] and references to [13] given there. We conclude that $\Phi^{-1}(p) \cap \mathcal{M}_{\mathbb{R}}$ consists of $2^g \cdot 2^{d-1} \cdot d!$ distinct points for every $p \in \mathcal{H}^\circ$.

Note also that, even if p is hyperbolic, it will typically admit real determinantal representations $\det(tM_1 + yM_2 + zM_3)$ that are not definite, i.e. are not equivalent

to such a representation with $M_1 = I_d$ and M_2, M_3 real. Such representations do not reflect the hyperbolicity of p .

2. NUIJ-PATH

In order to use homotopy continuation methods for numerical computations, we need an explicit path connecting any two given points in the space \mathcal{H} of hyperbolic polynomials. Following Nuij [9], we consider the operators

$$\begin{aligned} T_{k,s}: p &\mapsto p + sx_k \frac{\partial p}{\partial t} \\ G_s: p &\mapsto p(t, sx) \\ F_s &= T_{1,s}^d \cdots T_{n,s}^d. \end{aligned}$$

Proposition 2.1 (Nuij [9]). *For $p \in \text{int}(\mathcal{H})$, we have $H_s(p) = F_{1-s}G_s(p) \in \text{int}(\mathcal{H})$ for all $s \in [0, 1]$, with $F_0G_1(p) = p$ and $F_1G_0(p)$ not depending on p . \square*

We call $[0, 1] \ni s \mapsto H_s(p)$ the N -path of p . In order to ensure smoothness of the map $\Phi: \mathcal{M} \rightarrow \mathcal{H}$ along an N -path, we would like to ensure that the N -path stays inside the set \mathcal{H}° of smooth hyperbolic polynomials and thus away from the ramification locus.

Proposition 2.2. *Let $d = 2$. Then $H_s(p) \in \mathcal{H}^\circ$ holds for all $p \in \mathcal{H}$ and all $s \in [0, 1]$.*

Proof. Let $D = \text{Diag}(d_1, d_2)$ and $R = \begin{bmatrix} r_{(1,1)} & r_{(1,2)} \\ r_{(1,2)} & r_{(2,2)} \end{bmatrix}$. The quadric

$$H_s(\Phi(D, R)) = Ax^2 + Bxy + Cy^2 + Dxt + Eyt + Ft^2$$

is not contained in \mathcal{H}° if and only if it factors, which happens if and only if

$$\begin{aligned} 4 \text{Disc}(H_s(\Phi(D, R))) &= 4 \det \begin{bmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{bmatrix} \\ &= 2s^2(s-1)^2(d_1 - d_2)^2 + \\ &\quad 2s^2(s-1)^2(r_{11} - r_{22})^2 + \\ &\quad s^4r_{12}^2(d_1 - d_2)^2 + \\ &\quad 8r_{12}^2s^2(1-s)^2 + \\ &\quad 16(s-1)^4 \\ &= 0. \end{aligned}$$

The sum-of-squares representation was produced using the intuition obtained by a numerical sum-of-squares decomposition delivered by YALMIP and further exact symbolic computations in MATHEMATICA (see `Nuij-d2.mathematica` [8]). The components of the sum-of-squares decomposition above vanish simultaneously only when $s = 1$ (this proves the proposition) and either $r_{12} = 0$ or $d_2 = d_1$. \square

Remark 2.3. There is numerical evidence that in case $d = 3$ the N-path stays away from the locus singular cubics.

An attempt to plug in $H_s(\Phi(D, R))$ into the discriminant freezes all computer algebra systems: the result, a polynomial of degree 36 in 9 variables a_{ij} and d_i , is an extremely long expression if written in the monomial basis.

While it seems infeasible to derive symbolically that $(\mathcal{H} \setminus \mathcal{H}^\circ) \cap \{\text{N-path}\}$ has no real points except for $s = 1$ in a way similar to Proposition 2.2, our experiments driven by numerical real optimization in MATHEMATICA support this conclusion (see Nuij-d3.mathematica [8]).

In view of the preceding proposition, it seems natural to ask the following.

Question 2.4. It is true for all p that the N-path $H_s(p)$, with $s \in [0, 1)$, does not meet the ramification locus of the cover map Φ ?

Should the answer be no, one can still attempt to avoid the ramification points in the following manner.

Proposition 2.5. *The N-path $H_s(p)$, with parameter s varied along a piecewise-linear path $[0, c] \cup [c, 1) \subset \mathbb{C}$, does not meet the ramification locus of Φ , for almost all $c \in \mathbb{C}$. (More precisely, this holds for any c taken in the complement of some proper real algebraic subset of $\mathbb{C} \simeq \mathbb{R}^2$.)*

Proof. This immediately follows from the fact that the ramification locus is a proper complex subvariety of \mathcal{F} and therefore has real codimension at least 2. \square

Remark 2.6. Note that using a random path as described in the Proposition may result in a non-real determinantal representation: indeed, a non-real path for s is not guaranteed to result in a real point in the fiber $\Phi^{-1}(H_0(p))$ when a real point in $\Phi^{-1}(H_1(p))$ is taken. Nevertheless, the probabilistic algorithm suggested by the Proposition gives a way to construct not only complex, but real determinantal representations.

One can ask for the probability of obtaining a real determinantal representation at the end of the path described in Proposition 2.5. (For a more precise question, one may pick c on a unit circle with a uniform distribution.) While this probability is clearly non-zero, deriving an explicit lower bound seems to be a very hard problem. A naive intuition suggests that the probability can be estimated as a ratio of the number of real representations (Remark 1.6) to the total count of complex representations (Theorem 1.3). The experiments with quartic hyperbolic curves, however, suggest that the probability is much higher.

3. ALGORITHM AND IMPLEMENTATION

Given a hyperbolic polynomial $p \in \mathcal{H}$, the N-path $H_s(p)$ connects $p = H_1(p)$ with $p_0 = H_0(p)$ which does not depend on p . This suggests the following algorithm to compute a determinantal representation $(D_p, R_p) \in \mathcal{M}_{\mathbb{R}}$ for p :

- (1) Pick $(D_q, R_q) \in \mathcal{M}_{\mathbb{R}}$ giving a strictly hyperbolic polynomial $q = \Phi(D_q, R_q)$. Track the homotopy path $H_s(q)$ from $s = 1$ with the *start* solution (D_q, R_q) to $s = 0$ producing the *target* solution (D_{p_0}, R_{p_0}) . Then $p_0 = \Phi(D_{p_0}, R_{p_0})$.

- (2) Track the homotopy path $H_s(p)$ from $s = 0$ with the start solution (D_{p_0}, R_{p_0}) to $s = 1$ to obtain (D_p, R_p) such that $p = \Phi(D_p, R_p)$.

Note that, in principle, the first step only has to be performed once in each degree d . In what follows we describe two ways to set up a polynomial homotopy continuation for the pullback of an N-path.

3.1. N-path in the monomial basis. One way is to take the coefficients of the polynomial $\Phi(D, R) - H_s(p) \in \mathbb{C}[D, R, s][x, y, t]$ with respect to the monomial basis of \mathcal{F} . This gives a family of square ($\# \text{equations} = \# \text{unknowns}$) systems of polynomial equations in $\mathbb{C}[D, R]$ parametrized by s .

Then this family is passed to a homotopy continuation software package (we use NAG4M2 [7]). As long as $s \in \mathbb{C}$ follows a path that ensures that $H_s(p)$ stays in $\mathcal{H}^\circ \subset \mathcal{H}$, there are no singularities on the homotopy path except, perhaps, at the target system (see discussion in Section 2).

The bottleneck of this approach is the expansion of the determinant in the expression $\Phi(D, R)$ and evaluation of its (x, y, t) -coefficients: it takes $\Theta(d!)$ operations and results in the expression with $\Theta(d!)$ terms. This limits us to $d \leq 5$ in the current implementation of this approach.

3.2. N-path with respect to a dual basis. While it may seem that picking a basis of \mathcal{F} different from the monomial one does not bring any advantage, it turns out to be crucial for practical computation in case of larger d .

We fix a *dual basis* in \mathcal{F}^* consisting of $m = \dim \mathcal{F}$ evaluations e_i at general points $(x_i, y_i, t_i) \in \mathbb{C}^3$, for $i = 1, \dots, m$. The current implementation generates the points with coordinates on the unit circle in \mathbb{C} at random.

Now the family of polynomial systems to consider is

$$h_i = e_i(\Phi(D, R) - H_s(p)) \in \mathbb{C}[D, R, s], \quad i = 1, \dots, m.$$

Since $e_i(\Phi(D, R)) = \det(It_i + Dx_i + Ry_i)$, the evaluation of h_i and its partial derivatives costs $O(d^3)$. We modified the NAG4M2 implementation of evaluation circuits (a.k.a., straight-line programs) to include taking a determinant as a basic evaluation node.

4. EXAMPLE

The last improvement in the implementation allows us to compute examples for larger d . With an implementation of the homotopy tracking in arbitrary precision arithmetic, we see no obstacles to computing determinantal representations for d in double digits.

To give an example, we choose the sextic

$$\begin{aligned} p = & -36x^6 - 157x^4y^2 - 20x^3y^3 - 109x^2y^4 + 246xy^5 - 92y^6 - 12x^3y^2t + 90x^2y^3t \\ & + 10xy^4t + 76y^5t + 49x^4t^2 + 156x^2y^2t^2 - 16xy^3t^2 + 132y^4t^2 + 12xy^2t^3 \\ & - 14y^3t^3 - 14x^2t^4 - 27y^2t^4 + t^6. \end{aligned}$$

The polynomial p is hyperbolic, since $p = \Phi(D, R)$ with

$$(4.1) \quad D = \text{Diag}(-3, -2, -1, 1, 2, 3), \quad R = \begin{bmatrix} 0 & 1 & -1 & 1 & 2 & 1 \\ 1 & 0 & -1 & -2 & 1 & -1 \\ -1 & -1 & 0 & 1 & 2 & 1 \\ 1 & -2 & 1 & 0 & -1 & 1 \\ 2 & 1 & 2 & -1 & 0 & -2 \\ 1 & -1 & 1 & 1 & -2 & 0 \end{bmatrix}.$$

Assuming this pair (D, R) is not known, let us describe the application of our algorithm to recover a determinantal presentation of p ; one can reproduce the following results by running lines in `showcase.m2` [8]. First, taking an arbitrary pair (D_q, R_q) and tracking the N-path $H_s(q)$ from the strictly hyperbolic polynomial $q = \Phi(D_q, R_q) = H_1(q)$ to the fixed polynomial $p_0 = H_0(q)$ we get

$$D_{p_0} = \text{Diag}(.222847, 1.18893, 2.99274, 5.77514, 9.83747, 15.9829)$$

$$R_{p_0} = \begin{bmatrix} 6 & 2.51352 & 1.19571 & 4.04309 & 1.42786 & -1.98597 \\ 2.51352 & 6 & 3.08656 & .468873 & 2.38468 & 1.05948 \\ 1.19571 & 3.08656 & 6 & .785785 & 4.66027 & 2.29433 \\ 4.04309 & .468873 & .785785 & 6 & 1.6226 & .933245 \\ 1.42786 & 2.38468 & 4.66027 & 1.6226 & 6 & 3.50198 \\ -1.98597 & 1.05948 & 2.29433 & .933245 & 3.50198 & 6 \end{bmatrix}$$

Tracking the N-path $H_s(p)$ from $p_0 = H_0(p) = H_0(q)$ to $p = H_1(p)$, we obtain

$$D' = \text{Diag}(-3, -2, -, 1, 1, 2, 3)$$

$$R' = \begin{bmatrix} 0 & .596508 & -1.43241 & 2.00316 & 1.10471 & -.725394 \\ .596508 & 0 & .739773 & 1.79407 & .0604427 & -1.60948 \\ -1.43241 & .739773 & 0 & 1.56816 & 1.66137 & -.165953 \\ 2.00316 & 1.79407 & 1.56816 & 0 & .839374 & 2.00885 \\ 1.10471 & .0604427 & 1.66137 & .839374 & 0 & 1.57679 \\ -.725394 & -1.60948 & -.165953 & 2.00885 & 1.57679 & 0 \end{bmatrix}$$

which is an alternative determinantal presentation of p . While we returned to the same point p in the base of the cover Φ , the route taken has led us to a different sheet than the sheet of the fiber point $(D, R) \in \Phi^{-1}(p)$ in (4.1) used to construct this example.

With the default settings of NAG4M2 the homotopy tracking algorithm takes 28 steps on the first path and 15 steps on the second.

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